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Effect of automorphisms on variation of Hodge structures

By

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Introduction

Let X be a smooth projective variety with the ample canonical invertible sheaf K_X defined over C . Then, the Kuranishi family $\pi: \mathcal{X} \rightarrow S$ of the deformations of $\pi^{-1}(s_0) = X$ ($s_0 \in S$) is canonically polarized and universal, and hence $\text{Aut}(X)$ induces an action on the family $\pi: \mathcal{X} \rightarrow S$ preserving s_0 . Take $\sigma \in \text{Aut}(X)$, set $S^\sigma = \{\text{the fixed points of } \sigma \text{ in } S\}$ and denote by $\pi^\sigma: \mathcal{X}^\sigma \rightarrow S^\sigma$ the restriction of the family $\pi: \mathcal{X} \rightarrow S$ to $S^\sigma \hookrightarrow S$. Then σ induces an action on the variation $H^\sigma = (H^\sigma_Z, V^\sigma, F^\sigma, Q^\sigma)$ of polarized Hodge structures of weight n arising from the restricted family $\pi^\sigma: \mathcal{X}^\sigma \rightarrow S^\sigma$. In particular, the local system $H^\sigma_C = H^\sigma_Z \otimes C$ (resp. each Hodge filter $(F^\sigma)^i$) decomposes $H^\sigma_C = \bigoplus_\lambda H^\sigma_{\lambda, C}$ (resp. $(F^\sigma)^i = \bigoplus_\lambda (F^\sigma)^i_\lambda$) into the eigen subsheaves under the action of σ and we have

$$H^\sigma_{\lambda, C} \otimes \mathcal{O}_{S^\sigma} = (F^\sigma)_\lambda^0 \supset (F^\sigma)_\lambda^1 \supset \cdots \supset (F^\sigma)_\lambda^n \supset \{0\}$$

for each eigen value λ (see Theorem 1.4). In this manner, each automorphism of X imposes a restriction on the variation of Hodge structures. We state this fact in the section 1.

In the sections 2 and 3, we study, as an example, the surfaces with $p_g = c_1^2 = 1$ and K ample. We calculate all the automorphisms of these surfaces and determine explicitly the induced action of each automorphism on the variation $H^\sigma = (H^\sigma_Z, V^\sigma, F^\sigma, Q^\sigma)$ of polarized Hodge structures of weight 2 arising from the restricted family $\pi^\sigma: \mathcal{X}^\sigma \rightarrow S^\sigma$ (see Theorem 2.14) (The calculation is carried out in the section 3). After constructing the fine moduli $\tilde{\pi}: \tilde{\mathcal{X}} \rightarrow \tilde{M}$ of marked surfaces and period map $\tilde{\phi}: \tilde{M} \rightarrow D$, we rephrase mainly interesting part of the above result into the language of period map $\tilde{\phi}$ and we get that some automorphisms of the surfaces X give an effect on the period map $\tilde{\phi}$ to have positive dimensional fibres through the points corresponding to X (see Theorem 2.29).

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Notation and convention.

Every variety, in this paper, is defined over the field C of complex numbers.

For complex analytic manifold X ,

\mathcal{O}_X^1 = the sheaf of holomorphic 1-forms on X ,

$\mathcal{O}_X^r = \wedge^r \mathcal{O}_X^1$,

$K_X = \det \mathcal{O}_X^1$ and

T_X = the dual sheaf of \mathcal{O}_X^1 .

§ 1. General theory

Let X be a d -dimensional smooth projective variety and let $\pi: \mathcal{X} \rightarrow S$ be the Kuranishi family of the deformations of $\varepsilon: X \rightarrow X_{s_0} = \pi^{-1}(s_0)$ ($s_0 \in S$). We denote by $\text{Aut}(\mathcal{X}, S, \pi, s_0)$ the automorphisms of the family $\pi: \mathcal{X} \rightarrow S$ preserving the point $s_0 \in S$, and let

$$\varepsilon^*: \text{Aut}(\mathcal{X}, S, \pi, s_0) \rightarrow \text{Aut}(X)$$

be the homomorphism sending $\sigma \in \text{Aut}(\mathcal{X}, S, \pi, s_0)$ to $\varepsilon^{-1} \circ (\sigma|_{X_{s_0}}) \circ \varepsilon \in \text{Aut}(X)$.

We assume, for simplicity, the following two conditions throughout this section:

(1.1) *The canonical invertible sheaf K_X of X is ample.*

(1.2) *The parameter space S is smooth.*

Lemma 1.3.

(1.3.1) *The family $\pi: \mathcal{X} \rightarrow S$ is canonically polarized.*

(1.3.2) *$\text{Aut}(X)$ is a finite group.*

(1.3.3) *$\varepsilon^*: \text{Aut}(\mathcal{X}, S, \pi, s_0) \rightarrow \text{Aut}(X)$ is an isomorphism.*

Proof. Since we consider the family $\pi: \mathcal{X} \rightarrow S$ in the sense of germ at s_0 and since ampleness is an open condition, (1.3.1) follows from (1.1).

X is canonically polarized and hence $\text{Aut}(X)$ is an algebraic group. By the vanishing theorem of Kodaira-Nakano, $H^0(X, T_X) = 0$, since $T_X \simeq \mathcal{O}_X^{d-1} \otimes K_X^{-1}$ and (1.1). Therefore we have (1.3.2).

$H^0(X, T_X) = 0$ implies that the Kuranishi family $\pi: \mathcal{X} \rightarrow S$ has the universal property (cf. [9]). (1.3.3) is an immediate consequence of this universality.

Q.E.D.

Let $H = (H_Z, \nabla, F, Q)$ be the variation of polarized Hodge structures of weight n over S arising from the canonically polarized family $\pi: \mathcal{X} \rightarrow S$ (cf. [2], [4]). We recall here briefly the notation H_Z, ∇, F and Q . Denote by $\omega \in H^0(S, R^2\pi_*Z)$ the cohomology class of the relative canonical invertible sheaf $K_{\mathcal{X}/S}$. Define

$$P^n\pi_*Q = \text{Ker}(R^n\pi_*Q \xrightarrow{\omega^{d-n+1} \wedge} R^{2d-n+2}\pi_*Q) \quad \text{and} \\ P^n\pi_*Z = P^n\pi^*Q \cap \text{Im}(R^n\pi_*Z \rightarrow R^n\pi_*Q).$$

Then, we denote

- by H_Z = the local system $P^n\pi_*Z$,
- by ∇ = the Gauss-Manin connection on $H_O = H_Z \otimes \mathcal{O}_S$,
- by F = the Hodge filtration of H_O and
- by Q = the locally constant bilinear form on H_O defined by

$$Q(\xi, \eta) = (-1)^{n(n-1)/2} \int_{X_s} \xi \wedge \eta \wedge \omega(s)^{d-n}$$

for $\xi, \eta \in P^n(X_s, \mathbb{C}) = H_O(s)$ ($s \in S$), where $X_s = \pi^{-1}(s)$ and $\omega(s) \in H^{1,1}(X_s)$ induced from ω .

Now we consider the effect of an automorphism of X on the variation of polarized Hodge structure H . $\text{Aut}(X)$ acts on the family $\pi: \mathcal{X} \rightarrow S$ via (1.3.3). Take $\sigma \in \text{Aut}(X)$ and denote by S^σ the fixed points of σ in S . Note that S^σ is a submanifold of S because σ is of finite order. Let

$$\pi^\sigma: \mathcal{X}^\sigma \rightarrow S^\sigma$$

be the restriction of the family $\pi: \mathcal{X} \rightarrow S$ to over S^σ and let $H^\sigma = (H_Z^\sigma, \nabla^\sigma, F^\sigma, Q^\sigma)$ be the variation of polarized Hodge structure arising from the restricted family $\pi^\sigma: \mathcal{X}^\sigma \rightarrow S^\sigma$. We see, by functoriality, that

H^σ = the restriction of H to S^σ .

Since σ induces the action on H^σ , in particular, the Hodge filtration

$$H_O^\sigma = (F^\sigma)^0 \supset (F^\sigma)^1 \supset \cdots \supset (F^\sigma)^n \supset \{0\}$$

is compatible with the action of σ on $H_O^\sigma = H_Z^\sigma \otimes \mathcal{O}_{S^\sigma}$. Let

$$H_O^\sigma = \bigoplus_\lambda H_\lambda^\sigma \quad (\text{resp. } (F^\sigma)^i = \bigoplus_\lambda (F^\sigma)^i)_\lambda$$

be the decomposition of the local system $H_O^\sigma = H_Z^\sigma \otimes \mathbb{C}$ (resp. the locally free sheaf $(F^\sigma)^i$) into the eigen subsystems H_λ^σ (resp. subsheaves $(F^\sigma)_\lambda^i$) under the action of σ , where λ denotes the corresponding eigen value.

Summarizing up the above, we can formulate the effect of an automorphism σ of X on the variation of polarized Hodge structures H as follows:

Theorem 1.4. *With the above notion, we have*

$$H_i^q \otimes \mathcal{O}_{S^\sigma} = (F^\sigma)_i^0 \supset (F^\sigma)_i^1 \supset \cdots \supset (F^\sigma)_i^n \supset \{0\}$$

for each eigen value λ .

Remark 1.5. Recall that the identification $T_S = R^1\pi_* T_{X/S}$ is compatible with the induced actions of σ . Let

$$T_S \otimes \mathcal{O}_{S^\sigma} = \bigoplus_\lambda T_\lambda$$

be the decomposition into the subsheaves under the action of σ . Then we have

$$T_{S^\sigma} = T_1,$$

that is, T_{S^σ} can be considered as the subsheaf of $R^1\pi_* T_{X^\sigma/S^\sigma}$ consisting of the σ -invariant sections.

§ 2. Example; surfaces with $p_g = c_1^2 = 1$ and K ample.

(a) F. Catanese showed in [1] that every canonical model of a minimal surface X with $p_g = c_1^2 = 1$ can be represented as a weighted complete intersection of type $(6, 6)$ in $P(1, 2, 2, 3, 3)$ (for the notion of weighted complete intersection see [7]). Note that if we assume furthermore the canonical invertible sheaf K_X to be ample, X has no rational curves with self-intersection number -2 and hence X is isomorphic to its canonical model.

Let $R = C[x_0, y_1, y_2, z_3, z_4]$ be the weighted polynomial ring with $\deg x_0 = 1$, $\deg y_1 = \deg y_2 = 2$ and $\deg z_3 = \deg z_4 = 3$. The defining equations of a smooth weighted complete intersection of type $(6, 6)$ in $P(1, 2, 2, 3, 3)$ can be normalized as follows (cf. [1]):

$$(2.1) \quad \begin{cases} f = z_3^2 + f^{(1)}z_4x_0 + f^{(3)}, \\ g = z_4^2 + g^{(1)}z_3x_0 + g^{(3)}, \end{cases}$$

where $f^{(1)}$ and $g^{(1)}$ are linear and $f^{(3)}$ and $g^{(3)}$ are cubic forms in x_0^2, y_1 and y_2 , i.e., by using the notation $y_0 = x_0^2$,

$$\begin{aligned} f^{(1)} &= \sum_{i=0}^2 f_i y_i, & f^{(3)} &= \sum_{0 \leq i \leq j \leq k \leq 2} f_{ijkl} y_i y_j y_k, \\ g^{(1)} &= \sum_{i=0}^2 g_i y_i, & g^{(3)} &= \sum_{0 \leq i \leq j \leq k \leq 2} g_{ijkl} y_i y_j y_k. \end{aligned}$$

These coefficients form a Zariski open set U in 26-dimensional affine space, that is,

$$U = \left\{ u \in A^{26} \left| \begin{array}{l} \text{the corresponding surface is a smooth} \\ \text{weighted complete intersection of type} \\ (6, 6) \text{ in } P(1, 2, 2, 3, 3) \end{array} \right. \right\}.$$

For $u, u' \in U$, denote by f and g (resp. f' and g') the normalized forms as (2.1) corresponding to u (resp. u') and by I_u (resp. $I_{u'}$) the homogeneous ideal of R generated by f and g (resp. f' and g'), and set $X_u = \text{Proj}(R/I_u)$ (resp. $X_{u'} = \text{Proj}(R/I_{u'})$). Since $K_{X_u} \simeq \mathcal{O}_{X_u}(1)$ (resp. $K_{X_{u'}} \simeq \mathcal{O}_{X_{u'}}(1)$), we have

$$\bigoplus_{m \geq 0} H^0(X_u, K_{X_u}^{\otimes m}) \simeq R/I_u \quad (\text{resp. } \bigoplus_{m \geq 0} H^0(X_{u'}, K_{X_{u'}}^{\otimes m}) \simeq R/I_{u'}).$$

Hence, an isomorphism $\sigma: X_u \rightarrow X_{u'}$ induces the automorphism as graded ring $\sigma: R \rightarrow R$ with $\sigma I_{u'} = I_u$ (we use the same letter σ for simplicity of notation). More explicitly, σ can be represented by a non-degenerate matrix

$$(2.2) \quad \begin{array}{|c|cc|} \hline 1 & & \\ \hline d_{10} & d_{11} & d_{12} \\ d_{20} & d_{21} & d_{22} \\ \hline & & d_3 \\ & & d_4 \\ \hline \end{array} \quad \text{or}$$

$$(2.3) \quad \begin{array}{|c|cc|} \hline 1 & & \\ \hline d_{10} & d_{11} & d_{12} \\ d_{20} & d_{21} & d_{22} \\ \hline & & d_3 \\ & & d_4 \\ \hline \end{array}$$

with the action

$$\begin{cases} \sigma x_0 = x_0, \\ \sigma y_i = d_{i0}x_0^2 + d_{i1}y_1 + d_{i2}y_2 \quad (i=1, 2), \\ \sigma z_i = d_i z_i \quad (i=3, 4), \end{cases}$$

in case (2.2), and

$$\begin{cases} \sigma x_0 = x_0, \\ \sigma y_i = d_{i0}x_0^2 + d_{i1}y_1 + d_{i2}y_2 \quad (i=1, 2), \\ \sigma z_3 = d_3 z_4, \\ \sigma z_4 = d_4 z_3, \end{cases}$$

in case (2.3)*)

*) σ can be represented in this manner by choosing a suitable pair of isomorphisms $K_{X_u} \simeq \mathcal{O}_{X_u}(1)$ and $K_{X_{u'}} \simeq \mathcal{O}_{X_{u'}}(1)$.

Denote by G the group consisting of these matrices σ . Then, the induced action of G on U is

$$f = \sigma f' / d_3^2, \quad g = \sigma g' / d_4^2$$

in case (2.2) and

$$g = \sigma f' / d_3^2, \quad f = \sigma g' / d_4^2$$

in case (2.3). Note that the quotient space U/G is the coarse moduli space of the surfaces with $p_g = c_1^2 = 1$ and K ample (cf. [11]).

Set $\pi': \mathcal{X}' \rightarrow U$ the smooth family of the weighted complete intersections of type (6, 6) in $\mathbf{P}(1, 2, 2, 3, 3)$ parametrized by U . The induced action of G on \mathcal{X}' is evident.

(b) Let X be a smooth weighted complete intersection of type (6, 6) in $\mathbf{P} = \mathbf{P}(1, 2, 2, 3, 3)$. Denote by ψ a basis of $H^0(X, K_X)$ and by C the divisor of the zeros of ψ , i.e. the canonical divisor of X . By using the well-known exact sequences

$$(2.4) \quad 0 \rightarrow T_X \rightarrow T_{\mathbf{P}} \otimes \mathcal{O}_X \rightarrow N_{X/\mathbf{P}} \rightarrow 0,$$

$$(2.5) \quad 0 \rightarrow \mathcal{O}_X \rightarrow \bigoplus_{0 \leq i \leq 4} \mathcal{O}_X(e_i) \rightarrow T_{\mathbf{P}} \otimes \mathcal{O}_X \rightarrow 0$$

(where $e_0 = 1$, $e_1 = e_2 = 2$ and $e_3 = e_4 = 3$) and

$$(2.6) \quad 0 \rightarrow \check{N}_{C/X} \rightarrow \mathcal{Q}_X^1 \otimes \mathcal{O}_C \rightarrow \mathcal{Q}_C^1 \rightarrow 0,$$

we can calculate easily the following data on cohomology groups:

$$(2.7) \quad H^0(X, T_X) = H^2(X, T_X) = 0, \quad \dim H^1(X, T_X) = 18.$$

$$(2.8) \quad H^0(X, \mathcal{Q}_X^1) = 0, \quad \dim H^1(X, \mathcal{Q}_X^1) = 19.$$

$$(2.9) \quad H^1(X, T_{\mathbf{P}} \otimes \mathcal{O}_X) = 0, \quad \dim H^1(X, T_{\mathbf{P}} \otimes K_X) = 1.$$

$$(2.10) \quad \dim H^0(C, \mathcal{Q}_X^1 \otimes \mathcal{O}_C) \leq 2.$$

Let ω be the fundamental (1, 1)-form on X corresponding to the canonical polarization of X and let

$$H^1(X, T_X \otimes K_X) \xrightarrow{\omega} H^2(X, K_X)$$

be the map defined as the contraction with ω . Tensoring K_X to the exact sequence (2.4) and taking the cohomology sequence, we have

$$H^0(X, N_{X/\mathbf{P}} \otimes K_X) \xrightarrow{\delta} H^1(X, T_X \otimes K_X) \rightarrow H^1(X, T \otimes K_X)$$

Lemma 2.11.

$$H^0(X, N_{X/\mathbf{P}} \otimes K_X) \xrightarrow{\delta} H^1(X, T_X \otimes K_X) \xrightarrow{\omega} H^2(X, K_X)$$

is exact.

Proof. $\omega \in H^1(X, \mathcal{O}_X^1)$ comes from some $\tilde{\omega} \in H^1(X, \mathcal{O}_P \otimes \mathcal{O}_X)$ and we have a canonical factorization

$$\begin{array}{ccc} H^1(X, T_X \otimes K_X) & \longrightarrow & H^2(X, K_X) \\ \downarrow & \nearrow \tilde{\omega} & \\ H^1(X, T_P \otimes K_X) & & \end{array}$$

Since ω is surjective and $\dim H^2(X, K_X) = \dim H^1(X, T_P \otimes K_X) = 1$ from (2.9), we get our assertion. Q.E.D.

(c) Let X be a surface with $p_g = c_1^2 = 1$ and K_X ample. By (2.7), we see that the Kuranishi family $\pi: \mathcal{X} \rightarrow S$ of the deformations of $\varepsilon: X \rightarrow X_{s_0}$, $= \pi^{-1}(s_0)$ ($s_0 \in S$) is a universal family with the smooth parameter space S of dimension 18. Let $H = (S, H_Z, \mathcal{F}, F, Q)$ be the variation of polarized Hodge structures of weight 2 arising from the family $\pi: \mathcal{X} \rightarrow S$.

Note that in case of weight 2, by virtue of the polarization Q , the Hodge filtration F can be uniquely determined by its second filter F^2 , i.e. $F^0 = H_{\mathbb{C}}$ and $F^1 = (F^2)^{\perp}$ with respect to the bilinear form Q . Note also that $\text{rank } F^0 = \dim P^2(X, \mathbb{C}) = 20$, $\text{rank } F^1 = \dim P^{2,0}(X) + \dim P^{1,1}(X) = 19$ and $\text{rank } F^2 = \dim P^{2,0}(X) = 1$. Hence, in order to get the explicit form of the result (1.4) for our present example, it is enough to perform the following program:

(2.12) Choose a representative from each equivalence class of

$$\left\{ \sigma \left| \begin{array}{l} \exists X: \text{a surface with } p_g = c_1^2 = 1 \text{ and } K_X \text{ ample,} \\ \text{s.t. } \sigma \in \text{Aut}(X) \end{array} \right. \right\} / \sim$$

where

$$\sigma \sim \sigma' \Leftrightarrow \exists \begin{cases} X, X': \text{surface with } p_g = c_1^2 = 1 \text{ and } K \text{ ample,} \\ \tau: X \xrightarrow{\sim} X', \end{cases}$$

s.t. $\sigma \in \text{Aut}(X)$, $\sigma' \in \text{Aut}(X')$ and $\sigma' = \tau \circ \sigma \circ \tau^{-1}$.

(2.13) For each representative σ in (2.12) and for each surface X with $\sigma \in \text{Aut}(X)$, determine explicitly the decompositions of the sheaves $H_{\mathbb{C}}^q$, $(F^q)^2$ and $T_S \otimes \mathcal{O}_{S\sigma}$ into their eigen subsheaves under the induced action of σ . (Here we use the notation $H_{\mathbb{C}}^q$, $(F^q)^2$ and $T_S \otimes \mathcal{O}_{S\sigma}$ in the same sense as in the section 1.)

We will carry out the above procedure in the next section. Consequently, we obtain:

Theorem 2.14. Any automorphism $\sigma \neq \text{id}$ of a complete, smooth

surface with $p_g = c_1^2 = 1$ and K ample is equivalent, in the sense of (2.12), to some σ_i in the table below and such a σ_i is uniquely determined by σ . The induced actions of σ on $T_S \otimes \mathcal{O}_{S\sigma}$, H_C^q and $(F^q)^2$ are as follows:

$\sigma \sim \sigma_i$	induced action of σ on $T_S \otimes \mathcal{O}_{S\sigma}$, $(F^q)^2$ and H^q respectively
$\sigma_1 = (1, 1, 1, 1, -1)$	$(I_{15}, -I_3) \quad (-1)$ $(-I_{16}, I_4)$
$\sigma_2 = (1, 1, -1, -1, i)$	$(I_7, -I_8, iI_2, -iI_1) \quad (-i)$ $(-iI_8, iI_8, I_3, -I_1)$
$\sigma_3 = (1, 1, 1, -1, -1)$	$(I_{12}, -I_6) \quad (1)$ $(I_{12}, -I_8)$
$\sigma_4 = (1, -1, -1, i, i)$	$(I_6, -I_6, iI_4, -iI_2) \quad (-1)$ $(-I_8, I_4, -iI_4, iI_4)$
$\sigma_5 = (1, -1, -1, i, -i)$	$(I_6, -I_6, iI_3, -iI_3) \quad (1)$ $(I_8, -I_4, iI_4, -iI_4)$
$\sigma_6 = (1, -i, i, \varepsilon, \varepsilon^{-1})$	$(I_2, -I_4, iI_3, -iI_3, \varepsilon I_1, \varepsilon^{-1} I_1, -\varepsilon I_2, -\varepsilon^{-1} I_2) \quad (1)$ $(I_4, -I_4, iI_2, -iI_2, \varepsilon I_2, \varepsilon^{-1} I_2, -\varepsilon I_2, -\varepsilon^{-1} I_2)$
$\sigma_7 = (1, -i, i, \varepsilon, -\varepsilon^{-1})$	$(I_2, -I_4, iI_3, -iI_3, \varepsilon I_2, \varepsilon^{-1} I_1, -\varepsilon I_1, -\varepsilon^{-1} I_2) \quad (-1)$ $(-I_4, I_4, -iI_2, iI_2, -\varepsilon I_2, -\varepsilon^{-1} I_2, \varepsilon I_2, \varepsilon^{-1} I_2)$
$\sigma_8 = (1, 1, \omega, 1, 1)$	$(I_9, \omega I_7, \omega^2 I_2) \quad (\omega)$ $(\omega I_9, \omega^2 I_9, I_2)$
$\sigma_9 = (1, 1, \omega, 1, -1)$	$(I_7, \omega I_6, -I_2, -\omega I_1, \omega^2 I_2) \quad (-\omega)$ $(-\omega I_7, -\omega^2 I_7, \omega I_2, \omega^2 I_2, -I_2)$
$\sigma_{10} = (1, 1, \omega, -1, -1)$	$(I_5, \omega I_5, -I_4, -\omega I_2, \omega^2 I_2) \quad (\omega)$ $(\omega I_5, \omega^2 I_5, -\omega I_4, -\omega^2 I_4, I_2)$
$\sigma_{11} = (1, \omega, \omega, 1, 1)$	$(I_6, \omega^2 I_4, \omega I_8) \quad (\omega^2)$ $(\omega^2 I_7, \omega I_7, I_6)$
$\sigma_{12} = (1, \omega, \omega, 1, -1)$	$(I_5, \omega^2 I_4, \omega I_6, -\omega I_2, I_1) \quad (-\omega^2)$ $(-\omega^2 I_6, -\omega I_6, -I_4, I_2, \omega^2 I_1, \omega I_1)$
$\sigma_{13} = (1, \omega, -\omega, -1, i)$	$(I_2, -\omega^2 I_2, -I_3, \omega^2 I_2, \omega I_3, -\omega I_3, i\omega I_1, -i\omega I_1, iI_1)$ $(-i\omega^2)$ $(-i\omega^2 I_3, i\omega I_3, i\omega^2 I_3, -i\omega I_3, -iI_2, iI_2, I_1,$ $-I_1, \omega^2 I_1, \omega I_1)$
$\sigma_{14} = (1, \omega, \omega, -1, -1)$	$(I_4, \omega^2 I_4, \omega I_4, -\omega I_4, -I_2) \quad (\omega^2)$ $(\omega^2 I_4, \omega I_4, I_2, -I_4, -\omega^2 I_2, -\omega I_2)$

$\sigma_{15} = (1, \omega, \omega^2, 1, 1)$	$(I_6, \omega I_0, \omega^2 I_0) \quad (1)$ $(I_{10}, \omega I_6, \omega^2 I_5)$
$\sigma_{16} = (1, \omega, \omega^2, 1, -1)$	$(I_5, -I_1, \omega I_5, \omega^2 I_5, -\omega I_1, -\omega^2 I_1) \quad (-1)$ $(-I_8, I_2, -\omega I_4, -\omega^2 I_4, \omega I_1, \omega^2 I_1)$
$\sigma_{17} = (1, \omega, \omega^2, -1, -1)$	$(I_4, -I_2, \omega I_4, \omega^2 I_4, -\omega I_2, -\omega^2 I_2) \quad (1)$ $(I_0, -I_4, \omega I_3, \omega^2 I_3, -\omega I_2, -\omega^2 I_2)$
$\sigma_{0'} = (1, 1, -1, (1, 1))$	$(I_9, -I_9) \quad (-1)$ $(-I_{11}, I_9)$
$\sigma_{3'} = (1, 1, -1, (1, -1))$	$(I_6, -I_6, iI_3, -iI_3) \quad (1)$ $(I_7, -I_5, iI_4, -iI_4)$
$\sigma_{4'} = (1, i, -i, (1, i))$	$(I_3, -I_3, iI_3, -iI_3, \varepsilon I_2, -\varepsilon^{-1} I_1, \varepsilon^{-1} I_1, -\varepsilon I_2) \quad (-i)$ $(-iI_4, iI_4, I_2, -I_2, \varepsilon^{-1} I_2, \varepsilon I_2, -\varepsilon I_2, -\varepsilon^{-1} I_2)$
$\sigma_{8'} = (1, -1, \omega^2, (1, 1))$	$(I_4, \omega^2 I_4, -I_5, -\omega^2 I_3, \omega I_1, -\omega I_1) \quad (-\omega^2)$ $(-\omega^2 I_5, -\omega I_5, \omega^2 I_4, \omega I_4, -I_1, I_1)$
$\sigma_{10'} = (1, -1, \omega^2, (1, -1))$	$(I_2, \omega^2 I_3, -I_3, -\omega^2 I_2, iI_2, -i\omega^2 I_1, -iI_2, i\omega^2 I_1,$ $\omega I_1, -\omega I_1) \quad (\omega^2)$ $(\omega^2 I_3, \omega I_3, -\omega^2 I_2, -\omega I_2, i\omega^2 I_2, -i\omega I_2, -i\omega^2 I_2,$ $i\omega I_2, I_1, -I_1)$

where we use the notation:

$\sigma \sim \sigma_i$ is the equivalence relation in (2.12).

$i = \sqrt{-1}$, $\omega = \exp(2\pi i/3)$ and $\varepsilon = \exp(2\pi i/8)$.

$$(1, d_1, d_2, d_3, d_4) = \begin{array}{|c|} \hline 1 \\ \hline d_1 \\ \hline d_2 \\ \hline d_3 \\ \hline d_4 \\ \hline \end{array} \in G \quad \text{and}$$

$$(1, d_1, d_2, (d_3, d_4)) = \begin{array}{|c|c|} \hline 1 & \\ \hline d_1 & \\ \hline d_2 & \\ \hline & d_3 \\ \hline & d_4 \\ \hline \end{array} \in G.$$

$(\lambda_1 I_{m_1}, \dots, \lambda_r I_{m_r})$ indicates that the rank of λ_i -eigen subsheaves is m_i ($i=1, \dots, r$).

Remark 2.15. *There are several relations among σ_i 's in the table in Theorem (2.14), e.g. $\sigma_7^4 = \sigma_6^4 = \sigma_5^2 = \sigma_4^2 = \sigma_3$, $\sigma_{16} = \sigma_8 \sigma_{11} \sigma_1$ etc. In particular, only the following are of prime order:*

$$\sigma_1, \sigma_3, \sigma_8, \sigma_{11}, \sigma_{16} \text{ and } \sigma_9.$$

Corollary 2.16. *For any surface X with $p_g = c_1^2 = 1$ and K_X ample,*

$$\text{Aut}(X) \rightarrow \text{Aut}(P^2(X, \mathbb{C}))$$

is injective.

Proof. This is an immediate consequence of Theorem 2.14. Q.E.D.

(d) In this subsection, we will rephrase some of the result in Theorem 2.14. We continue to use the notation $X, \pi: \mathcal{X} \rightarrow S, H = (S, H_{\mathbb{Z}}, \mathcal{V}, F, Q), \pi^s: \mathcal{X}^s \rightarrow S^s$ and $H^s = (S^s, H_{\mathbb{Z}}^s, \mathcal{V}^s, F^s, Q^s)$ in the same sense as in the subsection (c).

Let

$$(2.17) \quad \phi: S \rightarrow D$$

be the period map associating to the variation of polarized Hodge structures H . Recall that (2.17) is constructed in the following way: Fixing a C^∞ -trivialization of the family $\pi: \mathcal{X} \rightarrow S$, we get the isomorphisms $\alpha_s: P^2(X_s, \mathbb{C}) \rightarrow P^2(X, \mathbb{C})$ ($s \in S$) preserving the polarization Q . Then the map

$$\phi: S \rightarrow P^{19} = \{\text{lines in } P^2(X, \mathbb{C}) \text{ through the origin}\}$$

defined by

$$\phi(s) = \text{the line } \alpha_s(P^{2,0}(X_s)) \text{ in } P^2(X, \mathbb{C})$$

is holomorphic and factorizes

$$\begin{array}{ccc} S & \longrightarrow & P^{19} \\ & \searrow & \cup \\ & D & \subset \check{D} \end{array}$$

where

$$\begin{aligned} \check{D} &= \{\xi \in P^{19} \mid Q(\xi, \xi) = 0\} \quad \text{and} \\ D &= \{\xi \in \check{D} \mid Q(\xi, \bar{\xi}) > 0\}. \end{aligned}$$

This map $S \rightarrow D$ is the period map (2.17).

Lemma 2.18. *The fibre of the period map ϕ through s_0 is at most 2-dimensional.*

Proof. By the result of Griffiths ([3]), the differential $d\phi(s_0)$ of the period map ϕ at s_0 can be identified with the map

$$H^1(X, T_X) \rightarrow \text{Hom}(P^{2,0}(X), P^{1,1}(X))$$

induced from the pairing

$$T_X \otimes K_X \rightarrow \Omega_X^1.$$

On the other hand, we get the exact sequence

$$H^0(X, \Omega_X^1) \rightarrow H^0(X, \Omega_X^1 \otimes \mathcal{O}_C) \rightarrow H^1(X, T_X) \xrightarrow{\psi} H^1(X, \Omega_X^1)$$

where we use the notation ψ and C in the subsection (b). Since $H^0(X, \Omega_X^1) = 0$ (2.8), we have

$$\begin{aligned} \text{Ker } d\phi(s_0) &= \text{Ker}(H^1(X, T_X) \xrightarrow{\psi} H^1(X, \Omega_X^1)) \\ &\simeq H^0(X, \Omega_X^1 \otimes \mathcal{O}_C). \end{aligned}$$

Hence, we get the assertion from (2.10)

Q.E.D.

Proposition 2.19. *We use the notation in Theorem 2.14. If there exists $\sigma \in \text{Aut}(X)$ with $\sigma \sim \sigma_1$ or σ_8 (resp. $\sigma \sim \sigma_3$), then the fibre of the period map ϕ in (2.17) through s_0 is of dimension ≥ 1 (resp. $= 2$).*

Proof. Since \check{D} is a smooth quadratic hypersurface in P^{19} and D is an open subset of \check{D} in the classical topology, we see that T_D is a locally free sheaf of rank 18. On the other hand, the pullback of the horizontal tangent bundle T_D^h is $\text{Hom}(F^2, F^1/F^2)$ which is also of rank 18. Therefore we have

$$(2.20) \quad \phi^* T_D = \text{Hom}(F^2, F^1/F^2).$$

Note that, via the action on $P^2(X, \mathbb{C})$, $\text{Aut}(X)$ has the induced action on D and the period map ϕ in (2.17) becomes $\text{Aut}(X)$ -equivalent. Denote by D^σ the submanifold of D consisting of the fixed points of σ in D . Then, we have the commutative diagram

$$(2.21) \quad \begin{array}{ccc} S & \xrightarrow{\phi} & D \\ \cup & & \cup \\ S^\sigma & \xrightarrow{\phi^\sigma} & D^\sigma \end{array}$$

From (2.20) and the functoriality of variation of Hodge structures, we get

$$(\phi^\sigma)^*(T_D \otimes \mathcal{O}_{D^\sigma}) \simeq \phi^*(T_D) \otimes \mathcal{O}_{S^\sigma}$$

$$\begin{aligned} &\simeq \text{Hom}(F^2, F^1/F^2) \otimes \mathcal{O}_{S\sigma} \\ &\simeq \text{Hom}((F^\sigma)^2, (F^\sigma)^1/(F^\sigma)^2), \end{aligned}$$

Where the identification in every step is compatible with the action of σ .

By using the fact that the Hodge bundle $(F^\sigma)^0/(F^\sigma)^1$ can be identified with the complex conjugate of $(F^\sigma)^2$ and that σ induces a real operator on H^2_σ , we can derive the induced action of σ on $\text{Hom}((F^\sigma)^2, (F^\sigma)^1/(F^\sigma)^2)$ from the table in Theorem 2.14. Because of the same reason in Remark 1.5, $T_{D\sigma}$ can be naturally identified with the eigen subsheaf of $T_D \otimes \mathcal{O}_{D\sigma}$ with eigen value 1 under the action of σ . Thus, we get

(2.22)

	rank $T_{S\sigma}$	rank $T_{D\sigma}$
$\sigma \sim \sigma_1$	15	14
$\sigma \sim \sigma_3$	12	10
$\sigma \sim \sigma_8$	9	8

The assertion follows from (2.22) and (2.18).

Q.E.D.

Fix a smooth, complete surface X with $p_g = c_1^2 = 1$ and K_X ample and denote by L the Euclidian lattice consisting of the \mathbb{Z} -valued primitive cohomology group $P^2(X, \mathbb{Z})$ plus the Hodge-Riemann bilinear form Q on $P^2(X, \mathbb{Z})$. Recall that $\text{rank } P^2(X, \mathbb{Z}) = 20$ and the signature of Q is $(2, 18)$.

We use the notation in (a). Set

$$\tilde{U} = \{(u, \alpha) \mid u \in U, \alpha \in \text{Isom}(P^2(X_u, \mathbb{Z}), L)\},$$

Where $\alpha \in \text{Isom}(P^2(X_u, \mathbb{Z}), L)$ means an isomorphism as Euclidian lattices, i.e. an isomorphism of the \mathbb{Z} -modules compatible with the bilinear forms. By using the fundamental group $\pi_1(U)$ of U , we can define the topology on U so that the first projection

$$(2.23) \quad \nu: \tilde{U} \rightarrow U$$

becomes an étale covering. Let

$$\tilde{\pi}': \mathcal{X}' = \mathcal{X}' \times_U \tilde{U} \rightarrow \tilde{U}$$

be the base extension of the family $\pi': \mathcal{X}' \rightarrow U$ by the morphism (2.23). Then G has the induced actions on \tilde{U} and \mathcal{X}' , which make $\tilde{\pi}$ a G -equivariant map.

By a marked surface we understand a couple (X', α) consisting of a

smooth, complete surface X' with $p_i = c_i^2 = 1$ and an ample $K_{X'}$ and of an isomorphism $\alpha: P^2(X', \mathbb{Z}) \xrightarrow{\sim} L$ as Euclidian lattices. By a family of marked surfaces we mean a smooth, proper holomorphic map $f: Y \rightarrow Z$ of analytic spaces Y and Z with the property that every fibre of f is a marked surface, and we call the universal family among these families of marked surfaces the fine moduli of marked surfaces.

Proposition 2.24. *The quotient spaces $\tilde{M} = \tilde{U}/G$ and $\tilde{X} = \tilde{X}'/G$ have the structures of complex analytic manifold, and the family*

$$\tilde{\pi}: \tilde{X} = \tilde{X}'/G \rightarrow \tilde{M} = \tilde{U}/G$$

is the fine moduli of the marked surfaces with $\dim \tilde{M} = 18$.

Before proving the above proposition, we should prepare a lemma.

Lemma 2.25. *Let Y_i ($i=1, 2$) be topological spaces and let $f: Y_1 \rightarrow Y_2$ be a continuous map. Let G be a topological group and we consider the situation that G acts both on Y_i ($i=1, 2$) and, with these actions, f becomes a G -equivariant map. Then, if the action of G on Y_2 is proper, so is the action of G on Y_1 .*

Proof. Consider the commutative diagram

$$(2.26) \quad \begin{array}{ccc} G \times Y_1 & \xrightarrow{\Psi_1} & Y_1 \times Y_1 \\ id \times f \downarrow & & \downarrow f \times f \\ G \times Y_2 & \xrightarrow{\Psi_2} & Y_2 \times Y_2 \end{array}$$

where $\Psi_i(g, y_i) = (gy_i, y_i)$ for $g \in G$ and $y_i \in Y_i$ ($i=1, 2$). We must show that $\Psi_1^{-1}(K)$ is compact whenever K is a compact subset of $Y_1 \times Y_1$. We may assume without loss of generality that $K = K'' \times K'$ for compact subsets K' and K'' of Y_1 .

Restricting the diagram (2.26), we get

$$\begin{array}{ccc} G \times K' & \xrightarrow{\Psi'_1} & Y_1 \times K' \\ id \times f' \downarrow & & \downarrow f \times f' \\ G \times f(K') & \xrightarrow{\Psi'_2} & Y_2 \times f(K') \end{array}$$

Since Ψ'_2 is a proper map, so is Ψ'_1 . $id \times f'$ being also a proper map, we

see that the composite map $\Psi'_2 \circ (id \times f')$ is proper and consequently the map Ψ'_1 is proper. In particular, $\Psi_1^{-1}(K) = \Psi_1'^{-1}(K'' \times K')$ is compact. Q.E.D.

Proof of Proposition 2.24. Let

$$\Psi: G \times U \rightarrow U \times U$$

be the morphism defined by $\Psi(g, u) = (gu, u)$ for $g \in G$ and $u \in U$. Since Ψ is a morphism in the category of schemes, we can use the valuative criterion for showing the properness of the morphism Ψ . Let A be a discrete valuation ring and let K be its quotient field. Set $V = \text{Spec}(A)$ and $V' = \text{Spec}(K)$ and denote by η (resp. s) the generic point (resp. closed point) of V . Given a commutative diagram

$$(2.27) \quad \begin{array}{ccc} V' & \xrightarrow{\beta'} & G \times U \\ \downarrow & & \downarrow \Psi \\ V & \xrightarrow{\beta} & U \times U \end{array}$$

We must show existence and uniqueness of the morphism $\gamma: V \rightarrow G \times U$ which is compatible with the diagram (2.27).

Set $(\sigma_\eta, u_\eta) = \beta'(\eta)$ and

$$\begin{array}{ccc} \mathcal{X}'_i = \mathcal{X}' \times_V V & \longrightarrow & \mathcal{X}' \\ \pi'_i \downarrow & & \downarrow \pi' \\ V & \xrightarrow{pr_i \circ \beta} & U \end{array} \quad (i=1, 2),$$

where pr_i means the i -th projection of $U \times U$. Then, σ_η induces the isomorphism $X_{2,\eta} = \pi_2'^{-1}(\eta) \rightarrow X_{1,\eta} = \pi_1'^{-1}(\eta)$ as canonically polarized surfaces. Hence, by the theorem of Matsusaka-Mumford ([6]), there exists uniquely the isomorphism $\sigma: \mathcal{X}'_1 \rightarrow \mathcal{X}'_2$ over V which is the extension of σ_η . Considering this σ as a V -valued point of G , we get the desired morphism $\gamma: V \rightarrow G \times U$.

Combining the above result and Lemma 2.25, we see that the action of G on \tilde{U} and $\tilde{\mathcal{X}}'$ are proper, and hence the quotient spaces $\tilde{M} = \tilde{U}/G$ and $\tilde{\mathcal{X}} = \tilde{\mathcal{X}}'/G$ exist in the category of analytic spaces ([5]). According to Corollary 2.16, the actions of G on \tilde{U} and $\tilde{\mathcal{X}}'$ have no fixed points. Therefore, \tilde{M} and $\tilde{\mathcal{X}}$ are manifolds. The last part of the assertion is obvious from our construction. Q.E.D.

Let D be the classifying space, used in (2.17), with respect to the fixed

X. By using the fine moduli $\tilde{\pi}: \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{M}}$ obtained in Proposition 2.24, we can define the global period map

$$(2.28) \quad \emptyset: \tilde{\mathcal{M}} \rightarrow D$$

by $\emptyset(\tilde{m}) = (\text{the line } \alpha_{\tilde{m}}(P^{2,0}(X_{\tilde{m}})) \text{ in } L \otimes \mathbb{C})$ for $\tilde{m} \in \tilde{\mathcal{M}}$, where $\tilde{\pi}^{-1}(\tilde{m}) = (X_{\tilde{m}}, \alpha_{\tilde{m}})$.

For $\tilde{m} \in \tilde{\mathcal{M}}$ with $\tilde{\pi}^{-1}(\tilde{m}) = (X_{\tilde{m}}, \alpha_{\tilde{m}})$, set

$$\text{Aut}(X_{\tilde{m}}) = \left\{ \begin{array}{l} \text{the automorphisms of the surfaces } X_{\tilde{m}} \\ (\text{omitting the datum } \alpha_{\tilde{m}}) \end{array} \right\}$$

Using the notation in Theorem 2.14, define

$\tilde{\mathcal{M}}_i = \{\tilde{m} \in \tilde{\mathcal{M}} \mid \text{there exists } \sigma \in \text{Aut}(X_{\tilde{m}}) \text{ with } \sigma \sim \sigma_i\}$ for each σ_i in the table in Theorem 2.14.

After Remark 2.15, we are interested, in particular, in the automorphisms σ_i of prime order, that is,

$$\sigma_1, \sigma_3, \sigma_8, \sigma_{11}, \sigma_{15} \text{ and } \sigma_{0'}.$$

Note that σ_1 has the conjugate

$$\sigma_{1,2} = (1, 1, 1, -1, 1).$$

We denote σ_1 by $\sigma_{1,1}$ when we want to distinguish this from its conjugate $\sigma_{1,2}$. Using these conjugates, we have the relation

$$\sigma_3 = \sigma_{1,1}\sigma_{1,2},$$

Let $\tilde{p}: \tilde{U} \rightarrow \tilde{\mathcal{M}}$ be the projection (cf. Proposition 2.24) and let $\nu: \tilde{U} \rightarrow U$ be the covering (2.23). Set

$$\tilde{\mathcal{M}}_{1,j} = \tilde{p}(\nu^{-1}(\text{Fix}_U(\sigma_{1,j}))) \quad (j=1,2),$$

where $\text{Fix}_U(\sigma_{1,j})$ is the set of the fixed points of $\sigma_{1,j}$ in U . It is easy to see that $\tilde{\mathcal{M}}_i$ and $\tilde{\mathcal{M}}_{1,j}$ have the structures of analytic subspace of $\tilde{\mathcal{M}}$, and, in particular, $\tilde{\mathcal{M}}_3$ and $\tilde{\mathcal{M}}_{1,j}$ ($j=1,2$) are submanifolds.

Theorem 2.29. *With the above notation, we have:*

(2.29.1) $\dim \tilde{\mathcal{M}}_{1,j} = 15$ ($j=1,2$) and $\dim \tilde{\mathcal{M}}_3 = 12$. $\tilde{\mathcal{M}}_1 = \tilde{\mathcal{M}}_{1,1} \cup \tilde{\mathcal{M}}_{1,2}$ and $\tilde{\mathcal{M}}_{1,j}$ ($j=1,2$) intersect transversally with $\tilde{\mathcal{M}}_{1,1} \cap \tilde{\mathcal{M}}_{1,2} = \tilde{\mathcal{M}}_3$. For every point $\tilde{m} \in \tilde{\mathcal{M}}_1$ (resp. $\tilde{m} \in \tilde{\mathcal{M}}_3$), the fibre of the period map \emptyset in (2.28) through \tilde{m} is of dimension ≥ 1 (resp. $= 2$).

(2.29.2) $\dim \tilde{\mathcal{M}}_8 = 9$. For every point $\tilde{m} \in \tilde{\mathcal{M}}_8$, the fibre of \emptyset through \tilde{m} is of dimension ≥ 1 .

Proof. Take $\tilde{m} \in \tilde{\mathcal{M}}$ and $\tilde{u} \in \tilde{p}^{-1}(\tilde{m})$, and set $u = \nu(\tilde{u})$. Note, first, that $\nu: (\tilde{U}, \tilde{u}) \rightarrow (U, u)$ is isomorphic in the sense of germs and $(\tilde{\mathcal{M}}, \tilde{m})$ can be

considered as the parameter space of the Kuranishi family of the deformation of $X_{\tilde{m}}$. Hence, by Theorem 2.14, we get that

$$\dim \tilde{M}_{1,j} = 15 \quad (j=1, 2),$$

$$\dim \tilde{M}_3 = 12 \quad \text{and}$$

$$\dim \tilde{M}_8 = 9.$$

$\tilde{M}_1 = \tilde{M}_{1,1} \cup \tilde{M}_{1,2}$ is an immediate consequence of their definition.

Since $\text{Fix}_U(\sigma_{1,1})$ (resp. $\text{Fix}_U(\sigma_{1,2})$, $\text{Fix}_U(\sigma_3)$) is G -stable with the equations $f^{(w)}=0$ (resp. $g^{(w)}=0$, $f^{(w)}=g^{(w)}=0$) and $\tilde{\mathcal{P}}: \tilde{U} \rightarrow \tilde{M}$ is smooth, the assertion of $\tilde{M}_{1,1}$ and $\tilde{M}_{1,2}$ intersecting transversally with $\tilde{M}_{1,1} \cap \tilde{M}_{1,2} = \tilde{M}_3$ follows from the corresponding fact about $\text{Fix}_U(\sigma_{1,j})$ ($j=1, 2$) and $\text{Fix}_U(\sigma_3)$.

The statement about the dimension of the fibre of the period map ϕ is an interpretation of Proposition 2.19. Q.E.D.

Note 2.30. By using the method in the forthcoming paper ([8]), we can further observe that

$$\dim_{\tilde{m}} \phi^{-2}(\phi(\tilde{m})) = \begin{cases} 2 & \text{if and only if } \tilde{m} \in \tilde{M}_3 \text{ and} \\ 1 & \text{if } \tilde{m} \in \tilde{M}_1 \cup \tilde{M}_8 - \tilde{M}_3. \end{cases}$$

§ 3. Calculation

In this section, we solve the problems (2.12) and (2.13). We employ the notation of the previous section.

(a) As we mentioned in the section 2, (a), U and G have the following properties:

(3.1) For any surface X with $p_g=c_1^2=1$ and K_X ample, there exists $u \in U$, such that X is isomorphic to the weighted complete intersection X_u corresponding to u .

(3.2) Let $u, u' \in U$. Then, any isomorphism between X_u and $X_{u'}$, if exists, is induced from some element of G .

(3.3) For $u \in U$,

$$\text{Aut}(X_u) = \{\sigma \in G \mid \sigma u = u\}.$$

By these (3.1), (3.2) and (3.3), the problem (2.12) is divided into the following two elementary questions:

(3.4) Divide G into the conjugate classes with respect to the action of G on G itself as inner automorphism, and choose a representative from each conjugate class.

(3.5) Select those elements of G , from among the representatives obtained in (3.4), by which some point of U is fixed.

As for (3.4), after elementary calculation in linear algebra, we get:

Lemma 3.6. Any element of G can be normalized by the inner automorphism into one of the following matrices, which is uniquely determined up to the interchanges of d_1 and d_2 and of d_3 and d_4 :

$$(3.6.1) \quad \begin{array}{|c|} \hline 1 \\ \hline d_1 \\ d_2 \\ d_3 \\ d_4 \\ \hline \end{array}$$

$$(3.6.2) \quad \begin{array}{|c|c|} \hline 1 & \\ \hline d_1 & \\ d_2 & \\ \hline & 1 \\ & d_4 \\ \hline \end{array}$$

$$(3.6.3) \quad \begin{array}{|c|} \hline 1 \\ \hline d_1 \\ 1 \quad d_1 \\ d_3 \\ d_4 \\ \hline \end{array}$$

$$(3.6.4) \quad \begin{array}{|c|c|} \hline 1 & \\ \hline d_1 & \\ 1 \quad d_1 & \\ \hline & 1 \\ & d_4 \\ \hline \end{array}$$

$$(3.6.5) \quad \begin{array}{|c|} \hline 1 \\ \hline d_{10} \quad 1 \\ d_2 \\ d_3 \\ d_4 \\ \hline \end{array}$$

$$(3.6.6) \quad \begin{array}{|c|c|} \hline 1 & \\ \hline d_{10} \quad 1 & \\ d_2 & \\ \hline & 1 \\ & d_4 \\ \hline \end{array}$$

$$(3.6.7) \quad \begin{array}{|c|} \hline 1 \\ \hline \quad 1 \\ d_{20} \quad 1 \\ d_3 \\ d_4 \\ \hline \end{array}$$

$$(3.6.8) \quad \begin{array}{|c|c|} \hline 1 & \\ \hline \quad 1 & \\ d_{20} \quad 1 & \\ \hline & 1 \\ & d_4 \\ \hline \end{array}$$

$$(3.6.9) \quad \begin{array}{|c|} \hline 1 \\ \hline d_{10} \quad 1 \\ \quad 1 \quad 1 \\ d_3 \\ d_4 \\ \hline \end{array}$$

$$(3.6.10) \quad \begin{array}{|c|c|} \hline 1 & \\ \hline d_{10} \quad 1 & \\ \quad 1 \quad 1 & \\ \hline & 1 \\ & d_4 \\ \hline \end{array}$$

As an answer of the question (3.5), we get the following:

Proposition 3.7. *Those 22 matrices σ_i 's appeared in the table in Theorem 2.14 form a complete system of representatives of the equivalence classes in (2.12), and any two of these σ_i 's are not equivalent to each other.*

Proof. The proof consists of several steps.

Step 1. Since $\text{Aut}(X_u)$ is a finite group for every $u \in U$ by (2.7), we know that, among the canonical forms in Lemma 3.6, only the forms (3.6.1) and (3.6.2) can occur as automorphisms of X_u for some $u \in U$ and, a priori, we also know that every d_i of these matrices must be a root unity.

Step 2. Take $u \in U$ and let f and g be normalized forms (2.1) of defining equations of X_u . If $f_{111} = g_{111} = 0$ or $f_{222} = g_{222} = 0$, X_u would have points which lie on the singular locus of $\text{Proj}(R)$. Hence, we have that

$$(3.8) \quad \begin{cases} f_{111} \text{ or } g_{111} \text{ is not zero} & \text{and} \\ f_{222} \text{ or } g_{222} \text{ is not zero.} \end{cases}$$

If $f_1 = f_{111} = f_{112} = 0$, X_u would have the singular points with $x_0 = y_2 = z_3 = 0$. Similar reasoning shows that

$$(3.9) \quad \begin{cases} f_1, f_{111} \text{ or } f_{112} \text{ is not zero,} \\ f_2, f_{122} \text{ or } f_{222} \text{ is not zero,} \\ g_1, g_{111} \text{ or } g_{112} \text{ is not zero} & \text{and} \\ g_2, g_{122} \text{ or } g_{222} \text{ is not zero.} \end{cases}$$

If $f_0 = f_{001} = f_{002} = f_{000} = g_{000} = 0$, X_u would have the singular points with $y_1 = y_2 = z_3 = z_4 = 0$. Therefore, we see that

$$(3.10) \quad f_0, f_{001}, f_{002}, f_{000} \text{ or } g_{000} \text{ is not zero.}$$

By using the symmetry among the coefficients of f and g caused by the actions of the matrices

$$\rho_1 = \begin{array}{|c|c|c|} \hline 1 & & \\ \hline & 1 & \\ \hline & 1 & \\ \hline & & 1 \\ \hline & & 1 \\ \hline \end{array}$$

and $\rho_2 =$

$$\begin{array}{|c|c|c|} \hline 1 & & \\ \hline & 1 & \\ \hline & 1 & \\ \hline & & 1 \\ \hline & & 1 \\ \hline \end{array}$$

it is enough to consider the following possibilities:

- (3.11.1) $f_{111}f_{222}g_1g_2f_0 \neq 0.$
 (3.11.2) $f_{111}f_{222}g_1g_2f_{001} \neq 0.$
 (3.11.3) $f_{111}f_{222}g_1g_2f_{000} \neq 0.$
 (3.11.4) $f_{111}f_{222}g_1g_2g_{000} \neq 0.$
 (3.11.5) $f_{111}f_{222}g_1g_{122} \neq 0.$
 (3.11.6) $f_{111}f_{222}g_1g_{222} \neq 0.$
 (3.11.7) $f_{111}f_{222}g_{111}g_{122}f_0 \neq 0.$
 (3.11.8) $f_{111}f_{222}g_{111}g_{122}f_{001} \neq 0.$
 (3.11.9) $f_{111}f_{222}g_{111}g_{122}f_{002} \neq 0.$
 (3.11.10) $f_{111}f_{222}g_{111}g_{122}f_{000} \neq 0.$
 (3.11.11) $f_{111}f_{222}g_{111}g_{122}g_{000} \neq 0.$
 (3.11.12) $f_{111}f_{222}g_{111}g_{222}f_0 \neq 0.$
 (3.11.13) $f_{111}f_{222}g_{111}g_{222}f_{001} \neq 0.$
 (3.11.14) $f_{111}f_{222}g_{111}g_{222}f_{000} \neq 0.$
 (3.11.15) $f_{111}f_{222}g_{112}g_{122} \neq 0.$
 (3.11.16) $f_{111}g_{222}f_2g_1 \neq 0.$
 (3.11.17) $f_{111}g_{222}f_2g_{112} \neq 0.$
 (3.11.18) $f_{111}g_{222}f_{122}g_{112}f_0 \neq 0.$
 (3.11.19) $f_{111}g_{222}f_{122}g_{112}f_{001} \neq 0.$
 (3.11.20) $f_{111}g_{222}f_{122}g_{112}f_{002} \neq 0.$
 (3.11.21) $f_{111}g_{222}f_{122}g_{112}f_{000} \neq 0.$

Step 3. Let $\sigma = (1, d_1, d_2, d_3, d_4)$ be a matrix of the form (3.6.1). The condition $\sigma u = u$ means explicitly the following relations: We use the notation $d_0 = 1$.

$$(3.12) \quad \begin{cases} f_i d_i d_4 = f_i d_3^2 & (0 \leq i \leq 2), \\ f_{ijk} d_i d_j d_k = f_{ijk} d_3^2 & (0 \leq i \leq j \leq k \leq 2), \\ g_i d_i d_3 = g_i d_4^2 & (0 \leq i \leq 2) \text{ and} \\ g_{ijk} d_i d_j d_k = g_{ijk} d_4^2 & (0 \leq i \leq j \leq k \leq 2). \end{cases}$$

Now we can proceed case by case.

Case (3.11.1). From (3.12), we have the relations

$$d_1^3 = d_2^3 = d_3^2, \quad d_1 d_3 = d_2 d_3 = d_4^2 \quad \text{and} \quad d_4 = d_3^2.$$

Hence $\sigma = (1, r^3, r^3, r, r^2)$, where $r^7 = 1$. Suppose $r \neq 1$, then we get $g_{111} = g_{112}$

$=g_{122}=g_{222}=0$ from (3.12). But this implies that X_u contains those points $x_0=z_3=z_4=0$ which are singular points of $\text{Proj}(R)$. Therefore, in this case, only $\sigma=(1, 1, 1, 1, 1)$ occurs.

Case (3.11.2). From (3.12), we have

$$d_1^2=d_2^2=d_3^2, \quad d_1d_3=d_2d_3=d_4^2 \quad \text{and} \quad d_1=d_3^2.$$

Hence $\sigma=(1, \gamma^4, \gamma^4, \gamma^8, \gamma)$, where $\gamma^8=1$. Suppose $\gamma^2 \neq 1$, then we get $g_{111}=g_{112}$, $g_{122}=g_{222}=0$ from (3.12). This is impossible as in case (3.11.1). Therefore, in this case, we would have $\sigma=(1, 1, 1, 1, 1)$ or $(1, 1, 1, 1, -1)$.

We omit here such kind of routine argument for other cases (3.11.i) ($3 \leq i \leq 21$). As a result, in case of diagonal matrices, we would obtain

$$\sigma_1, \dots, \sigma_{17}$$

in the table in Theorem (2.14).

Step 4. We deal, in this step, with a matrix $\sigma=(1, d_1, d_2, (1, d_4))$ of the form (3.6.2). Note that, in case σ is an isotropy of some point u of U , $\sigma^2=(1, d_2^2, d_1^2, d_4, d_4)$ must be also an isotropy of the same point u . Therefore, after the result in Step 3, we may only consider the cases

$$\sigma^2=\sigma_i \quad (i=0, 3, 4, 8, 10, 11, 14, 15, 17),$$

where $\sigma_0=(1, 1, 1, 1, 1)$.

Case $\sigma^2=\sigma_0$. Considering the conjugates by ρ_1 , we have three possibilities:

$\sigma=(1, 1, 1, (1, 1))$, $(1, 1, -1, (1, 1))$ or $(1, -1, -1, (1, 1))$. In case $\sigma=(1, 1, 1, (1, 1))$ or $(1, -1, -1, (1, 1))$, we get $f^{(3)}(0, y_1, y_2)=\pm g^{(3)}(0, y_1, y_2)$, but this implies that X_u contains singular points of $\text{Proj}(R)$. Therefore, in this case, only $\sigma_0=(1, 1, -1, (1, 1))$ would occur.

Case $\sigma^2=\sigma_4$. By the same argument as above, we would have $\sigma_4=(1, i, -i, (1, i))$.

Case $\sigma^2=\sigma_8$. We have four possibilities:

$$\begin{aligned} \sigma &= (1, 1, \omega^2, (1, 1)), \quad (1, 1, -\omega^2, (1, 1)), \\ & (1, -1, \omega^2, (1, 1)) \quad \text{or} \quad (1, -1, -\omega^2, (1, 1)). \end{aligned}$$

In case $\sigma=(1, 1, \omega^2, (1, 1))$ or $(1, -1, -\omega^2, (1, 1))$, we have $f^{(3)}(0, y_1, y_2)=\pm g^{(3)}(0, y_1, y_2)$, which is impossible as before. In case $\sigma=(1, 1, -\omega^2, (1, 1))$, f and g must be

$$\begin{cases} f = z_3^2 + f_1 z_4 x_0 y_1 + f_0 z_4 x_0^3 + f_{111} y_1^3 + f_{222} y_2^3 + f_{011} x_0^2 y_1^2 + f_{001} x_0^4 y_1 + f_{000} x_0^6, \\ g = z_4^2 + f_1 z_3 x_0 y_1 + f_0 z_3 x_0^3 + f_{111} y_1^3 - f_{222} y_2^3 + f_{011} x_0^2 y_1^2 + f_{001} x_0^4 y_1 + f_{000} x_0^6, \end{cases}$$

and hence

$$f-g=(z_3-z_4)(z_3+z_4+f_1 x_0 y_1+f_0 x_0^3)+2f_{222} y_2^3, \quad \text{which shows that } X_u \text{ has}$$

the singular points with

$$z_3 - z_4 = z_3 + z_4 + f_1 x_0 y_1 + f_0 x_0^3 = y_2 = 0.$$

Therefore, in this case, only $\sigma_8 = (1, -1, \omega^2, (1, 1))$ would occur.

Case $\sigma^2 = \sigma_{10}$. By the similar reasoning as above, we would get $\sigma_{10} = (1, -1, \omega^2, (1, -1))$.

In a similar way as in the above cases, we can prove that there are no isotropies σ of some $u \in U$ in case $\sigma^2 = \sigma_i$ ($i = 11, 14, 15, 17$).

Step 5. Finally, we claim that every σ_i obtained in Step 3 and Step 4 really occurs. It is easy to prove, by Jacobian criterion, that, for general choice of the coefficients, the following equations define smooth weighted complete intersections of type $(6, 6)$ in $\mathbf{P}(1, 2, 2, 3, 3)$:

$$(3.13) \quad \begin{cases} f = z_3^2 + f_{111} y_1^3 + f_{222} y_2^3 + f_{000} x_0^6, \\ g = z_4^2 + g_{111} y_1^3 + g_{222} y_2^3 + g_{000} x_0^6. \end{cases}$$

$$(3.14) \quad \begin{cases} f = z_3^2 + f_{111} y_1^3 + f_{122} y_1 y_2^2 + f_{002} x_0^4 y_2, \\ g = z_4^2 + g_{112} y_1^2 y_2 + g_{222} y_2^3 + g_{001} x_0^4 y_1. \end{cases}$$

$$(3.15) \quad \begin{cases} f = z_3^2 + f_{111} y_1^3 + f_{122} y_1 y_2^2 + f_{000} x_0^6, \\ g = z_4^2 + g_0 z_3 x_0^3 + g_{112} y_1^2 y_2 + g_{222} y_2^3. \end{cases}$$

$$(3.16) \quad \begin{cases} f = z_3^2 + f_{111} y_1^3 + f_{222} y_2^3 + f_{011} x_0^2 y_1^2 + f_{001} x_0^4 y_1 + f_{000} x_0^6, \\ g = z_4^2 - f_{111} y_1^3 + f_{222} y_2^3 + f_{011} x_0^2 y_1^2 - f_{001} x_0^4 y_1 + f_{000} x_0^6. \end{cases}$$

$$(3.17) \quad \begin{cases} f = z_3^2 + f_{111} y_1^3 + f_{112} y_1^2 y_2 + f_{122} y_1 y_2^2 + f_{222} y_2^3 + f_{001} x_0^4 y_1 + f_{002} x_0^4 y_2, \\ g = z_4^2 - i f_{111} y_1^3 + i f_{112} y_1^2 y_2 - i f_{122} y_1 y_2^2 + i f_{222} y_2^3 + i f_{001} x_0^4 y_1 - i f_{002} x_0^4 y_2. \end{cases}$$

Giving an order by inclusion to the set consisting of the fixed points loci in U of σ_i 's the minimal members are those corresponding to

$$\sigma_2, \sigma_6, \sigma_{13}, \sigma_{14}, \sigma_{17}, \sigma_{17'}, \text{ and } \sigma_{10'}.$$

The point of U corresponding to (3.13) (resp. (3.14), (3.15), (3.16), (3.17)) is fixed by σ_{14} and σ_{17} (resp. σ_6 , σ_2 and σ_{13} , $\sigma_{10'}$, $\sigma_{17'}$).

Remark 3.18. *As we have already used in step 5 of the proof of Proposition (3.7), we can get easily the defining equations of the fixed points loci in U of σ_i 's in the table in Theorem 2.14, which are all linear.*

(b) let σ_i be one of the matrices in the table in Theorem 2.14 and let $u \in U$ be a point with $\sigma_i u = u$. Set $X = X_u$.

Proposition 3.19. *Each σ_i induces on $T_s \otimes \bigoplus_{\mathcal{S}^i}$ the action indicated in the table in Theorem (2.14).*

Proof. Note first that, in order to determine the induced action of σ_i on the locally free sheaf $T_s \otimes \mathcal{O}_{S^{\sigma_i}}$, it is enough to investigate the induced action of σ_i on its fibre $(T_s \otimes \mathcal{O}_{S^{\sigma_i}})_{(s_0)} \simeq H^1(X, T_X)$ at s_0 .

Since the morphisms in the exact sequence (2.4) are equivariant with respect to the induced actions of $\text{Aut}(X)$, so is the morphisms in the exact sequence

$$(3.20) \quad 0 \rightarrow H^0(X, T_P \otimes \mathcal{O}_X) \rightarrow H^0(X, N_{X/P}) \rightarrow H^1(X, T_X) \rightarrow 0,$$

where we use (2.7) and (2.9). Hence we can reduce the study of the induced action of $\sigma_i \in \text{Aut}(X)$ on $H^1(X, T_X)$ to that on $H^0(X, T_P \otimes \mathcal{O}_X)$ and $H^0(X, N_{X/P})$.

Denote by $\text{res } H^0(X, T_P \otimes \mathcal{O}_X)$ (resp. $\text{res } H^0(X, N_{X/P})$) the image of $H^0(X, T_P \otimes \mathcal{O}_X)$ (resp. $H^0(X, N_{X/P})$) by the restriction map to the open subset of X defined by $x_0 \neq 0$.

Now the proof of Proposition 3.19 will be accomplished in a sequence of lemmas.

Lemma 3.21. *We can choose as a \mathbb{C} -linear basis of $\text{res } H^0(X, T_P \otimes \mathcal{O}_X)$ the following:*

$$\left\{ (a/x_0^2) \frac{\partial}{\partial (y_i/x_0^2)} \mid \begin{array}{l} a \text{ is a monomial in } R \text{ of} \\ \text{degree } 2, i=1, 2 \end{array} \right\} \\ \cup \left\{ (a/x_0^3) \frac{\partial}{\partial (z_i/x_0^3)} \mid \begin{array}{l} a \text{ is a monomial in } R \text{ of} \\ \text{degree } 3, i=3, 4 \end{array} \right\}.$$

Proof. Let $q: A \rightarrow P$ be the principal G_m -bundle over $P = P(1, 2, 2, 3, 3)$. Recall that the exact sequence (2.5) is derived from the exact sequence

$$0 \rightarrow T_{A/P} \rightarrow T_A \rightarrow q^* T_P \rightarrow 0$$

by taking its direct image, taking G_m -invariant subsheaves and finally restricting to X , that is,

$$(3.22) \quad 0 \rightarrow (q_* T_{A/P})^{G_m} \otimes \mathcal{O}_X \rightarrow (q_* T_A)^{G_m} \otimes \mathcal{O}_X \rightarrow T_P \otimes \mathcal{O}_X \rightarrow 0.$$

Taking the cohomology sequence of (3.22), we have

$$H^0(X, (q_* T_A)^{G_m} \otimes \mathcal{O}_X) \xrightarrow{\tau} H^0(X, T_P \otimes \mathcal{O}_X) \rightarrow 0.$$

Note that the morphism τ above sends

$$\theta = a_0 \frac{\partial}{\partial x_0} + a_1 \frac{\partial}{\partial y_1} + a_2 \frac{\partial}{\partial y_2} + a_3 \frac{\partial}{\partial z_3} + a_4 \frac{\partial}{\partial z_4} \in H^0(X, (q_* T_A)^{G_m} \otimes \mathcal{O}_X)$$

with $a_i \in (R/I)_{e_i}$ ($0 \leq i \leq 4$), to the induced operator $\tau(\theta) \in H^0(X, T_P \otimes \mathcal{O}_X)$ from \mathcal{O}_P to \mathcal{O}_X , that is,

$$\begin{aligned} \operatorname{res} \tau(\theta) &= \sum_{1 \leq i \leq 2} \theta(y_i/x_0^2) \frac{\partial}{\partial(y_i/x_0^2)} + \sum_{3 \leq i \leq 4} \theta(z_i/x_0^3) \frac{\partial}{\partial(z_i/x_0^3)} \\ &= \sum_{1 \leq i \leq 2} \left(\frac{a_i}{x_0^2} - \frac{2y_i a_0}{x_0^3} \right) \frac{\partial}{\partial(y_i/x_0^2)} + \sum_{3 \leq i \leq 4} \left(\frac{a_i}{x_0^3} - \frac{3z_i a_0}{x_0^4} \right) \frac{\partial}{\partial(z_i/x_0^3)} \end{aligned}$$

In particular,

$$(3.23) \quad \operatorname{res} \tau \left(x_0 \frac{\partial}{\partial x_0} \right) = \sum_{1 \leq i \leq 2} (-2y_i/x_0^2) \frac{\partial}{\partial(y_i/x_0^2)} + \sum_{3 \leq i \leq 4} (-3z_i/x_0^3) \frac{\partial}{\partial(z_i/x_0^3)}.$$

It is evident that we can take

$$(3.24) \quad \left\{ x_0 \frac{\partial}{\partial x_0} \right\} \cup \left\{ a \frac{\partial}{\partial y_i} \mid a \text{ is a monomial in } R \text{ of degree } 2, i=1, 2 \right\} \\ \cup \left\{ a \frac{\partial}{\partial z_i} \mid a \text{ is a monomial in } R \text{ of degree } 3, i=3, 4 \right\}$$

as a \mathbf{C} -linear basis of $H^0(X, (q^*T_{\mathcal{A}})^{\otimes m} \otimes \mathcal{O}_X)$. Combining (3.24), (3.23) and the fact $\dim \operatorname{Ker} \tau = 1$, we get the assertion. Q.E.D.

Lemma 3.25. *We can take as a \mathbf{C} -linear basis of $\operatorname{res} H^0(X, N_{X/P})$ the following:*

$$\left\{ (a/x_0^6) \frac{\partial}{\partial(f/x_0^6)} \mid a \text{ is a monomial in } R \text{ of degree } 6 \text{ except } z_3^2 \text{ and } z_4^2 \right\} \\ \cup \left\{ (a/x_0^6) \frac{\partial}{\partial(g/x_0^6)} \mid a \text{ is a monomial in } R \text{ of degree } 6 \text{ except } z_3^2 \text{ and } z_4^2 \right\}.$$

Proof. Under the well-known isomorphisms

$$H^0(X, N_{X/P}) \simeq H^0(X, \mathcal{O}_X(6))^{\oplus 2} \simeq (R/I)^{\oplus 2},$$

$(a, b) \in (R/I)^{\oplus 2}$ corresponds to the element $\gamma \in H^0(X, N_{X/P})$ with

$$\operatorname{res} \gamma = (a/x_0^6) \frac{\partial}{\partial(f/x_0^6)} + (b/x_0^6) \frac{\partial}{\partial(g/x_0^6)}.$$

We can exclude z_i^2 ($i=3, 4$) by using the relations of the ideal I . Q.E.D.

Lemma 3.26. *Let T (resp. N) be the \mathbf{C} -linear subspace of $\operatorname{res} H^0(X, T_P \otimes \mathcal{O}_X)$ (resp. $\operatorname{res} H^0(X, N_{X/P})$) spanned by*

$$\left\{ (y_i/x_0^2) \frac{\partial}{\partial(y_j/x_0^2)} \mid i=0, 1, 2; j=1, 2 \right\} \\ \cup \left\{ (z_i/x_0^3) \frac{\partial}{\partial(z_i/x_0^3)} \mid i=3, 4 \right\}$$

$$\begin{aligned}
& \left(\text{resp. } \left\{ (z_i x_0 y_i / x_0^6) \frac{\partial}{\partial (f/x_0^6)} \mid i=0, 1, 2 \right\} \right. \\
& \cup \left\{ (a/x_0^6) \frac{\partial}{\partial (f/x_0^6)} \mid a \text{ is a monomial in } y_i \right. \\
& \quad \left. (i=0, 1, 2) \text{ of degree } 6 \right\} \\
& \cup \left\{ (z_3 x_0 y_i / x_0^6) \frac{\partial}{\partial (g/x_0^6)} \mid i=0, 1, 2 \right\} \\
& \left. \cup \left\{ (a/x_0^6) \frac{\partial}{\partial (g/x_0^6)} \mid a \text{ is a monomial in } y_i \right. \right. \\
& \quad \left. \left. (i=0, 1, 2) \text{ of degree } 5 \right\} \right),
\end{aligned}$$

where we use the notation $y_0 = x_0^2$. Then, the sequence

$$0 \rightarrow T \rightarrow N \rightarrow \text{res } H^1(X, T_X) \rightarrow 0$$

induced from (3.20) is exact.

Proof. Recall that the morphism

$$\mu: \text{res } H^0(X, T_P \otimes \mathcal{O}_X) \rightarrow \text{res } H^0(X, N_{X/P})$$

sends

$$\sum_{1 \leq i \leq 2} (a_i/x_0^2) \frac{\partial}{\partial (y_i/x_0^2)} + \sum_{3 \leq i \leq 4} (a_i/x_0^3) \frac{\partial}{\partial (z_i/x_0^3)}$$

in $\text{res } H^0(X, T_P \otimes \mathcal{O}_X)$ to

$$\begin{aligned}
& \left\{ \sum_{1 \leq i \leq 2} (a_i/x_0^2) \frac{\partial (f/x_0^6)}{\partial (y_i/x_0^2)} + \sum_{3 \leq i \leq 4} (a_i/x_0^3) \frac{\partial (f/x_0^6)}{\partial (z_i/x_0^3)} \right\} \frac{\partial}{\partial (f/x_0^6)} \\
& + \left\{ \sum_{1 \leq i \leq 2} (a_i/x_0^2) \frac{\partial (g/x_0^6)}{\partial (y_i/x_0^2)} + \sum_{3 \leq i \leq 4} (a_i/x_0^3) \frac{\partial (g/x_0^6)}{\partial (z_i/x_0^3)} \right\} \frac{\partial}{\partial (g/x_0^6)}
\end{aligned}$$

in $\text{res } H^0(X, N_{X/P})$, and hence we have, in particular,

$$(3.27) \quad \begin{cases} \mu \left((a/x_0^3) \frac{\partial}{\partial (z_3/x_0^3)} \right) = 2(z_3 a/x_0^6) \frac{\partial}{\partial (f/x_0^6)} + (g^{(1)} x_0 a/x_0^6) \frac{\partial}{\partial (g/x_0^6)}, \\ \mu \left((a/x_0^3) \frac{\partial}{\partial (z_4/x_0^3)} \right) = (f^{(1)} x_0 a/x_0^6) \frac{\partial}{\partial (f/x_0^6)} + 2(z_4 a/x_0^6) \frac{\partial}{\partial (g/x_0^6)}, \end{cases}$$

where a stands for a monomial in R of degree 3. By the relations of the ideal I , we see, furthermore, that

$$(3.28) \quad \begin{cases} \mu \left((z_3/x_0^3) \frac{\partial}{\partial (z_3/x_0^3)} \right) = -2((f^{(1)} z_4 x_0 + f^{(3)})/x_0^6) \frac{\partial}{\partial (f/x_0^6)} \\ \quad + (g^{(1)} z_3 x_0/x_0^6) \frac{\partial}{\partial (g/x_0^6)}, \\ \mu \left((z_4/x_0^3) \frac{\partial}{\partial (z_4/x_0^3)} \right) = (f^{(1)} z_4 x_0/x_0^6) \frac{\partial}{\partial (g/x_0^6)} \\ \quad - 2((g^{(1)} z_3 x_0 + g^{(3)})/x_0^6) \frac{\partial}{\partial (f/x_0^6)}. \end{cases}$$

By (3.27) and (3.28), we can eliminate the members

$$\left\{ (z_3 a/x_0^6) \frac{\partial}{\partial (f/x_0^6)} \middle| \begin{array}{l} a \text{ is a monomial in } x_0, y_1, y_2 \\ \text{and } z_4 \text{ of degree 3} \end{array} \right\} \\ \cup \left\{ (z_4 a/x_0^6) \frac{\partial}{\partial (g/x_0^6)} \middle| \begin{array}{l} a \text{ is a monomial in } x_0, y_1, y_2 \\ \text{and } z_3 \text{ of degree 3} \end{array} \right\}$$

of the basis of $\text{res } H^0(X, N_{X/P})$ given in Lemma 3.25 and we obtain the assertion. Q.E.D.

Continuation of Proof of Proposition 3.19. By using the bases of $\text{res } H^0(X, T_P \otimes \mathcal{O}_X)$ and $\text{res } H^0(X, N_{X/P})$ given in Lemma 3.21 and Lemma 3.25 respectively, we can determine the induced action of σ_i on $\text{res } H^1(X, T_X)$ and hence, by the identity theorem, on $H^1(X, T_X)$. Lemma 3.26 contributes to save trouble in calculation. We add here a remark that, in case of σ_i ($i=0', 3', 4', 8', 10'$), we have to change the bases in Lemma 3.26 into more suitable ones, that is, the bases consisting of eigen vectors. The actual calculation is a routine task and we omit it. Q.E.D.

Proposition 3.29. *The induced action of each σ_i on $(F^{e_i})^2$ is as in the table in Theorem 2.14.*

Proof. As in the proof of Proposition 3.19, it is enough to study the induced action of σ_i on the fibre $(F^{e_i})^2(s_0) \simeq H^0(X, K_X)$ of the invertible sheaf $(F^{e_i})^2$ at s_0 . Let ψ be the global section of K_X corresponding to $x_0 \in R_1$ under the isomorphisms

$$H^0(X, K_X) \simeq H^0(X, \mathcal{O}_X(1)) \simeq (R/I)_1 \simeq R_1.$$

Then, by the Poincaré residue formula, we have

$$(3.30) \quad \text{res } \psi = (x_0/x_0) \left(\frac{\partial (f/x_0^6, g/x_0^6)}{\partial (z_3/x_0^3, z_4/x_0^3)} \right)^{-1} d(y_1/x_0^3) \wedge d(y_2/x_0^3),$$

where by res we mean the restriction to the open subset of X defined by $x_0 \neq 0$ and the Jacobian $\frac{\partial (f/x_0^6, g/x_0^6)}{\partial (z_3/x_0^3, z_4/x_0^3)} \neq 0$. Since $\text{res } \psi$ forms a basis of $\text{res } H^0(X, K_X)$, we can calculate, by (3.30), the induced action of σ_i on $\text{res } H^0(X, K_X)$, which determines that on $H^0(X, K_X)$ by the identity theorem. Q.E.D.

Proposition 3.31. *Each σ_i induces on $H^0_{\mathcal{O}}(F^{e_i})$ the action stated in the table in Theorem 2.14.*

Proof. As before, it is enough to investigate the induced action of σ_i

on the fibre $H^i_{\mathcal{G}}(s_0) \simeq P^2(X, \mathbf{C})$. By using the Hodge decomposition

$P^2(X, \mathbf{C}) = P^{2,0}(X) \oplus P^{1,1}(X) \oplus P^{0,2}(X)$ with $P^{0,2}(X) = \bar{P}^{2,0}(X)$ and the fact that σ_i induces a real operator on $P^2(X, \mathbf{C})$, we have already known, by Proposition 3.29, the induced action on $P^{2,0}(X)$ and $P^{0,2}(X)$.

The remaining thing is to determine the induced action of σ_i on $P^{1,1}(X)$.

Tensoring K_X to the exact sequence (2.4) and taking its cohomology sequence, we have the exact sequence

$$(3.32) \quad 0 \rightarrow H^0(X, T_{\mathbf{P}} \otimes K_X) \rightarrow H^0(X, N_{X/\mathbf{P}} \otimes K_X) \rightarrow P^{1,1}(X) \rightarrow 0$$

by (2.8) and Lemma 2.11. Note that the morphisms in the exact sequence (3.32) are all equivariant with respect to the induced actions of $\text{Aut}(X)$, and hence the problem is reduced to two parts, that is, determination of the induced actions on $H^0(X, T_{\mathbf{P}} \otimes K_X)$ and $H^0(X, N_{X/\mathbf{P}} \otimes K_X)$.

Since, in the rest part of the proof, the arguments are parallel to those in the proof of Proposition 3.19, we will only state the consequence of each step. By *res* we mean here the restriction to the open subset of X defined by $x_0 \neq 0$ and the Jacobian $\frac{\partial(f/x_0^6, g/x_0^6)}{\partial(z_3/x_0^3, z_4/x_0^3)} \neq 0$.

Lemma 3.33. *We can take as a \mathbf{C} -linear basis of $\text{res } H^0(X, T_{\mathbf{P}} \otimes K_X)$ the following:*

$$\begin{aligned} & \{(y_i/x_0^3)\theta \mid i=1, 2\} \\ & \cup \left\{ (a/x_0^3) \frac{\partial}{\partial(y_i/x_0^3)} \otimes \psi' \mid \begin{array}{l} a \text{ is a monomial in } R \text{ of} \\ \text{degree } 3, i=1, 2 \end{array} \right\} \\ & \cup \left\{ (a/x_0^4) \frac{\partial}{\partial(z_i/x_0^3)} \otimes \psi' \mid \begin{array}{l} a \text{ is a monomial in } R \text{ of} \\ \text{degree } 4, i=3, 4 \end{array} \right\}, \end{aligned}$$

where

$$\begin{aligned} \psi' &= \left(\frac{\partial(f/x_0^6, g/x_0^6)}{\partial(z_3/x_0^3, z_4/x_0^3)} \right)^{-1} d(y_1/x_0^2) \wedge d(y_2/x_0^2) \quad \text{and} \\ \theta &= - \left(\sum_{1 \leq i \leq 2} 2(y_i/x_0^2) \frac{\partial}{\partial(y_i/x_0^3)} + \sum_{3 \leq i \leq 4} 3(z_i/x_0^3) \frac{\partial}{\partial(z_i/x_0^3)} \right) \otimes \psi'. \end{aligned}$$

Lemma 3.34. *We can take as a \mathbf{C} -linear basis of $\text{res } H^0(X, N_{X/\mathbf{P}} \otimes K_X)$ the following:*

$$\begin{aligned} & \left\{ (a/x_0^7) \frac{\partial}{\partial(f/x_0^6)} \otimes \psi' \mid \begin{array}{l} a \text{ is a monomial in } R \text{ of degree} \\ 7 \text{ except } z_3^2 x_0 \text{ and } z_4^2 x_0 \end{array} \right\} \\ & \cup \left\{ (a/x_0^8) \frac{\partial}{\partial(g/x_0^6)} \otimes \psi' \mid \begin{array}{l} a \text{ is a monomial in } R \text{ of degree} \\ 7 \text{ except } z_3^2 x_0 \text{ and } z_4^2 x_0 \end{array} \right\} \end{aligned}$$

where we use the notation ψ' in Lemma 3.33.

Lemma 3.35. Let T' (resp. N') be the \mathbb{C} -linear subspace of $\text{res } H^0(X, T_{\mathbb{P}} \otimes K_X)$ (resp. $\text{res } H^0(X, N_{X/\mathbb{P}} \otimes K_X)$) spanned by

$$\begin{aligned} & \{(y_i/x_0^3)\theta \mid i=1, 2\} \\ & \cup \left\{ (a/x_0^3) \frac{\partial}{\partial (y_i/x_0^3)} \otimes \psi' \mid a \text{ is a monomial in } R \text{ of degree } 3, i=1, 2 \right\} \\ & \cup \left\{ (z_i x_0/x_0^4) \frac{\partial}{\partial (z_i/x_0^3)} \otimes \psi' \mid i=3, 4 \right\} \\ & \left(\text{resp. } \left\{ (z_i a/x_0^7) \frac{\partial}{\partial (f/x_0^6)} \otimes \psi' \mid a \text{ is a monomial in } x_0, y_1 \right. \right. \\ & \quad \left. \left. \text{and } y_2 \text{ of degree } 4 \right\} \right) \\ & \cup \left\{ (a/x_0^7) \frac{\partial}{\partial (f/x_0^6)} \otimes \psi' \mid a \text{ is a monomial in } x_0, y_1 \right. \\ & \quad \left. \text{and } y_2 \text{ of degree } 7 \right\} \\ & \cup \left\{ (z_3 a/x_0^7) \frac{\partial}{\partial (g/x_0^6)} \otimes \psi' \mid a \text{ is a monomial in } x_0, y_1 \right. \\ & \quad \left. \text{and } y_2 \text{ of degree } 4 \right\} \\ & \cup \left\{ (a/x_0^7) \frac{\partial}{\partial (g/x_0^6)} \otimes \psi' \mid a \text{ is a monomial in } x_0, y_1 \right. \\ & \quad \left. \text{and } y_2 \text{ of degree } 7 \right\} \Bigg\}. \end{aligned}$$

Then, (3.32) induces the exact sequence

$$0 \rightarrow T' \rightarrow N' \rightarrow \text{res } P^{1,1}(X) \rightarrow 0.$$

Continuation of Proof of Proposition 3.31. By using the above lemmas, we can calculate, as in the proof of Proposition 3.19, the induced action of σ_i on $P^{1,1}(X)$. Q.E.D.

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